MTH221, LINEAR ALGEBRA, FINAL EXAM FALL 2006

Question 1. *(**6 points**) Let* $A = \begin{bmatrix} 2 & 123 & -1 \\ 1 & 456 & 1 \\ 2 & 789 & 1 \end{bmatrix}$ *If you know that* $det(A) = -420$*, then find the value of* x_2 *in the solution of the linear system* $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Question 2. *Determine whether the following vectors are linearly independent in the vector space V* (**SHOW YOUR WORK**).

a)(4 points) $V = R^3$, $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

b) (4 points) $V = P_3, \quad 3x^2, \quad x^2 - 10x + 15, \quad 10x - 15$

Question 3. *For each of the following sets of vectors, determine if it is a subspace. If YES explain why and, if your answer is a NO then give an example to show why it is not subspace.*

*(a) (***5 points***)* $S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mid x_1 - 5x_3 = 0 \right\}$

*b)(***5 points***)* $S = \{B \in R^{2\times 2} \mid A \;\; \textit{is singular (non-invertible)}\ \}$

Question 4. *Find a Basis for each of the following subspace S of the given vector space V. (***DO NOT SHOW IT IS A SUBSPACE***)*

*a) (***5 points***)* $V = R^5$, $S = \left\{ \begin{bmatrix} a+b \\ b \\ c \\ 0 \\ c+b \end{bmatrix} \mid a, b, c \in R \right\}$

*b) (***5 points***)* $V = R^{2x2}$, $S = \{A \in R^{2\times 2} \mid -A = A^T\}$

*c)(***5 points***)* $V = P_4$, $S = \{p \in P_4 \mid p(1) = p(0) = 0\}$

Question 5. *Let* $S = Span \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix} \right\}$.

*a)(***4 points***) Find a basis for S.*

*b)(***4 points***) Find an ORTHONORMAL BASIS for S.*

Question 6. *Consider the following set of polynomials in* P_3.

$$S = \{t^2 + 1,\ t^2 + 2t,\ 3t^2 + t - 1\}$$

c. *(***4 points***) Find a basis for* $Span\{S\}$.

b. *(***4 points***) Does* $6t^2 - 1$ *belong to* $Span\{S\}$*?*

Question 7. *We have a* 3×3 *matrix* $A = \begin{bmatrix} a & 1 & 2 \\ b & 3 & 4 \\ c & 5 & 6 \end{bmatrix}$ *with* $det(A) = 3$. *Compute the determinant of the following matrices:*

*(a) (***2 points***)* $\begin{bmatrix} a-2 & 1 & 2 \\ b-4 & 3 & 4 \\ c-6 & 5 & 6 \end{bmatrix}$

*(b) (***2 points***)* $\begin{bmatrix} 7a & 7 & 14 \\ b & 3 & 4 \\ c & 5 & 6 \end{bmatrix}$

*(c) (***3 points***)* $2A^{-1}A^T$

*d) (***4 points***)* $\begin{bmatrix} a-2 & 1 & 2 \\ b & 3 & 4 \\ c & 5 & 6 \end{bmatrix}$

Question 8.

$$\textit{If } A = \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}$$

*a) (**4 points**) Find all eigenvalues A*

*b) (**6 points**)Find a nonsingular matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$ (that it is $A = QDQ^{-1}$)*

*c) (**5 points**) For the matrix A in Question number 8, find* A^5

Question 9. *(**6 points**) Let* $L : R^3 \longrightarrow R^3$ *be a linear transformation such that*

$$L\left(\begin{bmatrix}-2\\1\\-2\end{bmatrix}\right)=\begin{bmatrix}-3\\1\\2\end{bmatrix},\quad L\left(\begin{bmatrix}3\\2\\-1\end{bmatrix}\right)=\begin{bmatrix}2\\-2\\1\end{bmatrix},\quad L\left(\begin{bmatrix}-1\\-1\\1\end{bmatrix}\right)=\begin{bmatrix}-1\\2\\4\end{bmatrix}.$$

Find $L\left(\begin{bmatrix}1\\1\\-1\end{bmatrix}\right)$

Question 10. *It is given that* $A = \begin{bmatrix} 1 & 0 & -2 & 1 & 3 \\ -1 & 1 & 5 & -1 & -3 \\ 0 & 2 & 6 & 0 & 1 \\ 1 & 1 & 1 & 1 & 4 \end{bmatrix}$ *and its reduced row echelon form is given by* $B = \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) *(**2 points**) Find the rank of A.*

(b) *(**2 points**)Find the nullity of A.*

(c) *(**3 points**) Find a basis for the column space of A.*

(d) *(**3 points**) Find a basis for the row space of A.*

(e) *(**3 points**) Find a basis for the null space of A.*

Final Review

1. (10 points) Find the general solution of

$$\begin{aligned} x + 2y + z &= 1 \\ -x - 2y + z &= 2 \\ 2x + 4y + 2z &= 2 \end{aligned}$$

2. Find the inverse and the determinant of $A = \begin{bmatrix} -6 & 2 & -2 \\ 2 & -2 & 0 \\ -2 & 0 & 2 \end{bmatrix}$

3. Suppose A is a 3×3 invertible matrix and $A^{-1} = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 5 & 2 \\ 1 & 0 & 3 \end{bmatrix}$.

   (a) Solve the system of equations $AX = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

   (b) Solve the system of equations $A^T X = \begin{bmatrix} 2008 \\ 1 \\ 0 \end{bmatrix}$

4. Suppose A and B are 4x4 matrices. If $det(A) = -2$ and $det(B) = 3$ find

   (a) $det(3A^{-1}.B^{-1})$

   (b) $det(A^2.4B^{-1})$

   (c) $det(adj(A))$

5. Suppose A is a 2×2 invertible matrix. If the row operation $-2R_1 + R_2 \rightarrow R_2$ and then the row operation $-R_2 + R_1 \rightarrow R_1$ is performed on A, it becomes the identity matrix.

   (a) Find two elementary matrices E_1 and E_2 such that $E_2 E_1 A = I_{2\times 2}$.

   (b) Write A as a product of elementary matrices. [Hint: Use (a)]

   (c) Write A^{-1} as a product of elementary matrices. [Hint: Use (a)]

6. (10 points) For which values of x (if any) is the matrix $\begin{pmatrix} 1 & 0 & -3 \\ 0 & x & 2 \\ 3 & -10 & x \end{pmatrix}$ singular (not invertible)?

7. (12 points) Express

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 4 & 0 \\ 3 & 0 & 4 \end{pmatrix}$$

as a product of elementary matrices.

8. Find a basis for the subspace of R^4 spanned by

$$\{(2,9,-2,53),\ (-3,2,3,-2),\ (8,-3,-8,17),(0,-3,0,15)\}$$

9. Let $A = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 2 & 1 \\ 2 & -6 & 4 \end{bmatrix}$, find

   (a) the rank of the matrix

   (b) a basis for Nul(A).

   (c) a basis of the row space of A.

   (d) a basis for the column space of A.

10. Suppose $A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 3 & 5 & 2 & 3 \\ 1 & 0 & 3 & 2 \end{bmatrix}$ and echelon form of A is $\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

    (a) Find a basis for the column space of A.

    (b) Find the nullity of A.

    (c) Find a basis for the row space of A.

11. (12 points) Which of the following are subspaces of the given vector space V? Justify your answers.

    (a) $V = R^3$, $S = \{(x, y, 0) : x + y = 0\}$.

    (b) $V = R^3$, $S = \{(x, y, 0) : xy \geq 0\}$.

    (c) $V = R^{2\times 2}$, $S = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : \text{All matrices in } V \text{ where } abc = 0 \right\}$

12. (10 points) Find a Basis for each of the following subspaces S of the given vector space V.

    (a) $V = R^{2\times 2}$, $S = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : \text{All matrices in } V \text{ where } a + b - c = 0 \right\}$

    (b) $V = P_2$, $S = \{\ p(x)$ is in P_2 and $p(1) = 0\}$

13. (12 points)Determine whether the following sets of vectors are linearly independent in the vector space V. Justify your answers.

    (a) $V = R^3$, $v_1 = (1, 0, 0),\ v_2 = (1, 1, 1)\ , v_3 = (2, 2, 3)$.

    (b) $V = P_3$, $p_1(x) = x^2 + x - 1, p_2(x) = 1 - x^2, p_3(x) = x$

14. (10 points) Let T be the linear transformation from R^2 to R^2 given by $T(a,b) = (a-b, 2b+a)$.

   (a) Find the standard matrix representation of T.

   (b) Find $T^{-1}(x,y)$ if T^{-1} exists.

15. (14 points) Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

   (a) Find the characteristic Polynomial of A.

   (b) Hence find the Eigenvalues of A.

   (c) For each Eigenvalue of A, find a basis of the corresponding Eigenspace.

   (d) Decide if A is diagonalizable or not. Justify your answer. If yes, give an invertible matrix P and a diagonal matrix D such that $PDP^{-1} = A$.

16. (10 points) Consider the vectors $u = (a-1\ ,\ 1\ ,\ b)$, $v = (2\ ,\ a\ ,\ -1)$ and $w = (3\ ,\ a+b\ ,\ 2)$ in R^3. Find all values of a and b that make u orthogonal to both v and w.

17. Let $u_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$, $u_3 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$, and $x = \begin{bmatrix} 8 \\ -4 \\ -3 \end{bmatrix}$.

   (a) Show that the set $\beta = \{u_1, u_2, u_3\}$ is an orthogonal basis for R^3.

   (b) Express x as a linear combination of the elements in β.

18. Suppose $\beta = \left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -5 \\ 9 \\ -9 \\ 3 \end{bmatrix} \right\}$ is a basis for a subspace W. Use Gram-Schmidt to construct an orthonormal basis for W.

Final Exam for MTH 221 , Spring 2011

Ayman Badawi

QUESTION 1. (12pts, each = 1.5 points) Answer the following as true or false: NO WORKING NEED BE SHOWN.

(i) If A is a 3×3 matrix and det(A) = 4, then det(3A) = 12.

(ii) If A is a 10×10 matrix and det(A) = 2, then $det(AA^T) = 1$

(iii) If Q, F are independent points in R^n, then $Q.F = 0$ (Q.F means dot product of Q with F).

(iv) $T(a, b, c) = (2ab, -c)$ is a linear transformation from R^3 to R^2.

(v) If A is a 3×3 matrix and $det(A - \alpha I_3) = (1 - \alpha)^2(3 + \alpha)$ and $E_1 = span\{(2, 4, 0)\}$, then it is possible that A is diagnolizable.

(vi) If $T : R^2 \to R^2$ is a linear transformation and $Ker(T) = \{(0, 0)\}$, then T is onto.

(vii) If A is a 4×5 matrix, then dimension of N(A) is at least one.

(viii) If A is a 3×4 matrix and Rank(A) = 3, then the columns of A are dependent.

QUESTION 2. (8pts) For what value(s) of k is the system of equations below inconsistent?

$$\begin{aligned} -x + y + z &= k \\ 2x - 3y + z &= 2 \\ -y + kz &= 6 + k \end{aligned}$$

QUESTION 3. (i) **(5pts)**For which value(s) of x is the following matrix singular (non-invertible)?

$$\begin{pmatrix} 1 & x & 2 \\ -1 & 1 & 1 \\ -1 & 5 & x+1 \end{pmatrix}$$

(ii) **(5pts)**Find examples of 2×2 matrices A and B such that

$$\det(A) = \det(B) = 2 \text{ and } \det(A+B) = 25,$$

or explain why no such matrices can exist.

QUESTION 4. (12pts) Let

$$A = \begin{pmatrix} 2 & -1 & 0 \\ 1 & -1 & 0 \\ 2 & -2 & 3 \end{pmatrix}$$

(i) Find A^{-1}.

(ii) Use your result in (i) above to solve the system

$$\begin{aligned} 2x - y &= 1 \\ x - y &= 2 \\ 2x - 2y + 3z &= 1 \end{aligned}$$

(iii) Solve the system $(A^T)^{-1}X = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

(If you need more space, then use the back of this page)

QUESTION 5. (12pts)

(i) Form a basis, say B, for P_4 such that B contains the two independent polynomials : $f(x) = 1 + x + 2x^2, k(x) = -2 - 2x + x^2$.

(ii) Let $S = span\{(1, 1, -1, 0), (0, 1, 1, 1), (3, 5, -1, 2)\}$. Find an orthogonal basis for S.

(iii) Let S be the subspace as in (ii). Is $(2, 5, 1, 3) \in S$? EXPLAIN your answer.

QUESTION 6. (12pts)

(i) Let $S = \{(a, bc + a, c) \mid a, b, c \in R\}$. Is S a subspace of R^3? If yes, then find a basis for S. If No, then tell me why not.

(ii) Let $S = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in R \quad and \quad a + b + c = 0 \right\}$. Is S a subspace of $R^{2\times 2}$? If yes, then find a basis for S. If No, then tell me why not.

(iii) Let $S = \{f(x) \in P_4 \mid f(1) = 0 \quad \text{OR} \quad f(-2) = 0\}$. Is S a subspace of P_4? If yes, then find a basis for S. If No, then tell me why not.

(iv) $S = \left\{ \begin{bmatrix} x & -x \\ 1 & y \end{bmatrix} : x, y \epsilon R \right\}$. Is S a subspace of $R^{2\times 2}$. If yes, then find a basis. If No, then tell me why not.

QUESTION 7. (14pts) Let $T : R^4 \to R^3$ such that $T(a, b, c, d) = (a + 2b,\ \ -a - 2b + c - d,\ \ -2a - 4b - c + d)$ be a linear transformation.

(i) **(3pts)**Find the standard matrix representation of T, say M.

(ii) **(4pts)**Find a basis for Ker(T).

(iii) **(4pts)**Find a basis for the range of T.

(iv) **(3pts)**Is $(-2, 1, 3, 3) \in Kert(T)$? Explain

QUESTION 8. (8 pts) Let $T : P_2 \to R^2$ be a linear transformation such that $T(1+x) = (-6, -2)$, and $T(2-x) = (-3, -1)$

(i) Find T(5) and T(3x)

(ii) Is there a polynomial $f(x) = a + bx$ such that $T(a + bx) = (6, 2)$? if yes, then find such $f(x)$

QUESTION 9. (12pts) Given $A = \begin{bmatrix} 1 & 4 & 4 & 4 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$ is a diagonalizable matrix.

(i) Find a diagonal matrix D and an invertible matrix Q such that $A = QDQ^{-1}$.

(ii) Find A^{2012}.

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